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A FORMULA FOR THE IMAGINARY PARTS OF THE EIGENVALUES PROBLEM CONCERNING A SHELL OSCILLATION IN A COMPRESSIBLE FLUID*

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An exact formula connecting the imaginary parts of the eigenvalues of the problem of oscillation of a closed shell in an infinite compressible fluid, with the eigenfunctions of the problem possessing energy-type integrals, is obtained. Replacing these functions by the eigenfunctions of the problem for an incompressible fluid, yields an estimate.

In the problem of forced oscillations of a shell in a fluid, the resonance maxima are bounded and inversely proportional to the imaginary parts of the eigenvalues. Their estimate finds use when the effect of radiation is compared with other dissipative effects.

1. The system of equations of the problem of forced oscillations of a thin, elastic closed shell in an infinite perfect compressible fluid acted upon by an internal load Q, can be written in the form (with the factor $e^{-i\Omega t}$ deleted everywhere)

$$\mathbf{Lu} = \lambda(\mathbf{u} - \mathbf{A}\Phi) + \mathbf{Q} \quad (S) \tag{1.1}$$

$$\Phi = K(\omega)\varphi \quad (D \cup S),\ \varphi = K'(\omega)\varphi - w \tag{1.2}$$

$$\lambda = (c_1\omega/c_0)^2,\ \omega = \Omega R_0/c_1,\ \mathbf{A} = (0, 0, a),\ a = \rho_1 R_0/(\rho_0 h)$$

Here $\mathbf{L}$ is a differential, selfconjugate tensor-operator of the theory of shells, $\mathbf{u}$ is the shell displacement vector (w is its normal component), Φ is the displacement potential in the fluid, c is speed of sound, ρ is density (the indices 0 and 1 refer to the shell material and the fluid, respectively); R_0, h, S, D are the characteristic dimension, thickness, middle surface (the Liapunov surface) and the outside of the shell, respectivley; φ is the potential density, $K(\omega)$ and $K'(\omega)$ are the integral operators of the potential theory on S with kernels $(2\pi|\mathbf{x}-\mathbf{y}|)^{-1}\exp(i\omega|\mathbf{x}-\mathbf{y}|)$ and $k' = k_n$, respectively, where $\mathbf{x}, \mathbf{y}$ are the radius vectors of the observation and integration points, and the index n denotes differentiation along the outward normal to the surface (here towards S at the point $\mathbf{x}$). The corresponding homogeneous problem ($\mathbf{Q} = 0$) represents a nonlinear, nonself-conjugate eigenvalue problem (its solutions will be indicated by the degree sign $^\circ$).

In the case of an incompressible fluid ($c_1 \to \infty$: the Laplace equation replaces the Helmholtz equation in D and exp in the kernels k and k' is replaced by unity) the spectrum is discrete and distributed along the real axis /1/. Introduction of the compressibility shifts it into the complex plane /2/: $\lambda^\circ = \lambda + i\tau$, $\omega^\circ = \omega^* + i\varepsilon$.

Formal inspection of the solution of the eigenvalue problem uncovers the following characteristic feature. The sign of ε is such that Φ increase exponentially with respect to the space variables /3/

$$\Phi^\circ \sim |\mathbf{x}|^{-1}\exp[(i\omega^* - \varepsilon)|\mathbf{x}|],\ |\mathbf{x}| \to \infty\ (\varepsilon < 0)$$

It follows that the above solutions do not have a finite energy integral (class of solutions in $L_2(D \cup S)$ is almost empty: the exceptions consist of trivial cases of rotation of the shells of revolution as rigid units /2/). The following approach makes it possible to overcome the difficulties of the spectral analysis caused by the infinite character of the energy.

Use of the integral boundary equations (1.2) separates automatically the problem on S from the problem of continuing Φ into D. We can therefore limit ourselves to investigating a two-dimensional mathematical eigenvalue problem. This is sufficient for many cases (including the present). For example, solutions of (1.1) and (1.2) on S can be written in the form of series in eigenfunctions and Φ computed in D with help of the known quadrature method, using the result of expanding $\Phi|_S$ and $\Phi_n|_S = w$ as the starting expression.

Let us obtain a particular (auxilliary) solution of the inverse problem of forced oscillations. Using the given displacement vector $\mathbf{u} = \mathbf{u}^\circ$ and frequency $\omega = (c/c_1)\sqrt{\lambda}$ where ω corresponds to the projection of λ° on the real axis and $\mathbf{u}^\circ, \Phi^\circ, \lambda^\circ$ is a solution of the eigenvalue problem, and taking into account the remark made above, we obtain $\mathbf{Q}$ and Φ.

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Function Φ is given by the solution of the Neumann problem for the Helmholtz equation (I is a unit operator)

$$\Phi = K(\omega)\,[K'(\omega) - I]^{-1} w^\circ$$

We note that $\Phi|_S \neq \Phi^\circ$ because of the frequency shift from ω° to ω; Φ together with w satisfy (1.2) and, which is important, also $\Phi \in L_2(D \cup S)$.

Let us substitute $\mathbf{u}^\circ, \Phi, \omega$ into (1.1). Using the relation $\mathbf{Lu}^\circ = \lambda^\circ(\mathbf{u}^\circ - \mathbf{A}\Phi^\circ)$ we obtain

$$\mathbf{Q} = i\tau(\mathbf{u}^\circ - \mathbf{A}\Phi^\circ) - \mathbf{A}\lambda^\circ\delta\Phi \quad (\delta\Phi = \Phi^\circ - \Phi)$$

The above solution will be used later. Below we derive an energetic relationship.

2. Let us apply the Green's formula to the functions Φ and $\bar{\Phi}$ representing the solutions of (1.1) and (1.2) in the region bounded by the surfaces S and S_R (a sphere of large radius). Using the relations $(\Delta + \omega^2)\Phi = 0, \operatorname{Im}\omega = 0$ we obtain

$$\int_{S_R} (\bar{\Phi}\Phi_n - \Phi\bar{\Phi}_n)\, dS_R = \int_S (\cdots)\, dS = \Gamma \tag{2.1}$$

The passage to the limit $R \to \infty$, the Sommerfeld radiation condition written in integral form /4/ and the asymptotics of the solution at infinity, together make it possible to transform the left hand side of (2.1) to the form

$$\Gamma = 2i\omega(\Phi, \bar{\Phi})_\infty, \quad (\Phi, \bar{\Phi})_\infty = \lim_{R\to\infty} \int_{S_R} |\Phi|^2\, dS_R \tag{2.2}$$

The quantity Γ represents, with the accuracy of up to a constant multiplier, an energy flux emitted at infinity over a period.

Let us introduce the scalar product

$$(f, g) = \int_S fg\, dS$$

To transform the right-hand side of (2.1), we shall write the equation (1.1) twice, for $\mathbf{u}$ and Φ, and for the conjugated functions $\bar{\mathbf{u}}$ and $\bar{\Phi}$. Multiplying the first equation by $\bar{\mathbf{u}}$ and the second by $\mathbf{u}$, integrating each equation over the surface S and subtracting one from the other, we obtain

$$(\mathbf{Lu}, \bar{\mathbf{u}}) - (\mathbf{L}\bar{\mathbf{u}}, \mathbf{u}) = \lambda[(\mathbf{u}, \bar{\mathbf{u}}) - (\bar{\mathbf{u}}, \mathbf{u})] - \lambda a[(\Phi, \bar{w}) - (\bar{\Phi}, w)] + (\mathbf{Q}, \bar{\mathbf{u}}) - (\bar{\mathbf{Q}}, \mathbf{u}) \tag{2.3}$$

The left-hand side of (2.3) is identically equal to zero by virtue of the fact that the operator $\mathbf{L}$ is selfconjugate. Obviously, so is the first term of the right-hand side within the square bracket. It follows therefore from (2.1)—(2.3) that

$$(\bar{\mathbf{Q}}, \mathbf{u}) - (\mathbf{Q}, \bar{\mathbf{u}}) = 2i\omega\lambda a(\Phi, \bar{\Phi})_\infty \tag{2.4}$$

The above integral equation can be regarded as an equation of energy balance. The work done by the load $\mathbf{Q}$ shifted on $\mathbf{u}$ over one period is equal to the energy radiated over the same period (with the accuracy of up to a multiplier). Substituting the particular solution of Sect.1 into (2.4) yields an equation linear in τ the solution of which is

$$\frac{\tau}{\lambda} = -\frac{a\{\omega(\Phi, \bar{\Phi})_\infty - \operatorname{Im}(\delta\bar{\Phi}, w^\circ)\}}{(\mathbf{u}^\circ, \bar{\mathbf{u}}^\circ) - a\operatorname{Re}(\Phi^\circ, \bar{w}^\circ)} \tag{2.5}$$

The quantity τ in the exact formula (2.5) is expressed in terms of the integrals, which have the form of energy, of the natural oscillation modes and the function Φ derived from them, as well as $\operatorname{Re}\lambda^\circ$. This makes it possible to construct accurate estimates. Let us therefore substitute u° and ω by u' and ω', denoting the corresponding mode and frequency of oscillation of a shell in an incompressible fluid (by virtue of the separation of a part of the spectrum transforming into the spectrum of the simplified problem as $c_1 \to \infty$, see /2/). The limit integral $(\Phi, \Phi)_\infty$ can be written in terms of the potential densities φ using the formulas /4/

$$(\Phi, \Phi)_\infty = \int_0^{2\pi}\int_0^{\pi} |f(\theta, \psi)|^2 \sin\theta\, d\theta\, d\psi \equiv (f, f)_0 \tag{2.6}$$

$$f(\theta, \psi) = \frac{1}{2\pi}\int_S e^{i\omega\eta y}\varphi(y)\, dS, \quad \varphi = (K'(\omega) - I)^{-1} w^\circ$$

Here η denotes the unit vector from the center in the direction towards the observation point with coordinates $\sin\theta$, $\cos\psi$, $\sin\theta$, $\sin\psi$, $\cos\theta$.

Let us replace Φ° and φ in the formulas (2.5), (2.6) by the corresponding Φ' and φ' from the problem for an incompressible fluid. This yields the following approximate formula convenient for practical use:

$$\frac{\tau}{\lambda} = \frac{a\omega'(f', f')_0}{(u', u') - a(\Phi', w')} \tag{2.7}$$

$$f'(\theta, \psi) = \frac{1}{2\pi}\int\limits_S e^{i\omega'\eta y}\varphi'(y)\,dS, \quad \varphi' = (K'(0) - I)^{-1}w'$$

The formula (2.7) can be used to improve the solution of the forced problem for an incompressible fluid with poles at $\lambda = \lambda'$. Addition of the term $i\tau$ to λ', to account for radiation losses (other types of dissipation are dealt with in the same manner) removes these singularities and makes possible the estimation of the magnitudes of the resonance maxima.

Notes. 1). The operators K and K' on S have Taylor expansions in ω° and are symmetric when $\mathrm{Re}\,\omega^\circ = 0$. The general theorems on analyticity with respect to a parameter /5/ impy that the solutions taking into account and omitting the compressibility of the fluid, i.e. (2.5) and (2.7), differ from each other in the neighborhood of $\omega' = 0$ by the amount $\sim\omega'^2$ provided that they are equal to zero in the mean on S (in other words, they differ by $\sim\omega'$). From the formulas for the number of frequencies given in /2/ it follows that, at least for the shells of revolution and the series of frequencies of quasi-transverse oscillations the following asymptotic relation holds: $\mathrm{Re}\,\omega^\circ \sim h^\beta, h \to 0$ ($\beta > 0$ and $\beta = 1/2$ for the lower frequencies. The following assertion can be proved: (2.7) is an asymptotically exact formula for the given series, as $h \to 0$.

2). In practice it is often sufficient to estimate the order of magnitude of τ. In this case it is expedient to remember the criteria $\omega'/R_0 \ll 2\pi/l$ where l is the characteristic wave length in the shell. In this region of the spectrum the compressibility of the fluid has little effect on the solution in the neighborhood of the shell /6/, and it should be expected that (2.7) will provide a satisfactory approximation to the exact result.

3). Using the data given in /6/ for the eigenfunctions of the simplest shape shells (sphere, cylinder) we can obtain explicit formulas connecting τ with the number of the form, the frequency and the parameters of the shell and fluid.

4). Theory of oscillation of dissipative systems contains the formula

$$2\pi / q = \delta E / E \quad (\delta E \ll E)$$

where $q = \lambda/\tau$ is the Q factor, E is the total energy of the system and δE is the energy loss over a period. The formula (2.7) has the same meaning, since its denominator contains the total energy of the shell with the attached mass of fluid. In view of this it is clear that the problem of the finiteness of the energy touched upon in Sect.1 is far from trivial.

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